

# Hsu-Robbins and Spitzer's theorems for the variations of fractional Brownian motion

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## Abstract

Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein's method, we prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

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## 1 Introduction

A famous result by Hsu and Robbins [7] says that if  $X_1, X_2, \dots$  is a sequence of independent identically distributed random variables with zero mean and finite variance and  $S_n := X_1 + \dots + X_n$ , then

$$\sum_{n \geq 1} P(|S_n| > \varepsilon n) < \infty$$

for every  $\varepsilon > 0$ . Later, Erdős ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every  $\varepsilon > 0$  and  $X_1, X_2, \dots$  are independent and identically distributed, then  $\mathbf{E}X_1 = 0$  and  $\mathbf{E}X_1^2 < \infty$ . Since then, many authors extended this result in several directions.

Spitzer's showed in [13] that

$$\sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n) < \infty$$

for every  $\varepsilon > 0$  if and only if  $\mathbf{E}X_1 = 0$  and  $\mathbf{E}|X_1| < \infty$ . Also, Spitzer's theorem has been the object of various generalizations and variants. One of the problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as  $\varepsilon \rightarrow 0$  of the quantities

$\sum_{n \geq 1} P(|S_n| > \varepsilon n)$  and  $\sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n)$ . Heyde [5] showed that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sum_{n \geq 1} P(|S_n| > \varepsilon n) = \mathbf{E}X_1^2 \quad (1)$$

whenever  $\mathbf{E}X_1 = 0$  and  $\mathbf{E}X_1^2 < \infty$ . In the case when  $X$  is attracted to a stable distribution of exponent  $\alpha > 1$ , Spataru [12] proved that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{-\log \varepsilon} \sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n) = \frac{\alpha}{\alpha - 1}. \quad (2)$$

The purpose of this paper is to prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [5] or [12]. Recall that the fractional Brownian motion  $(B_t^H)_{t \in [0,1]}$  is a centered Gaussian process with covariance function  $R^H(t, s) = \mathbf{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$ . It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

$$V_n = \sum_{k=0}^{n-1} H_q \left( n^H \left( B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) \right) \quad (3)$$

where  $B$  is a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  (in the sequel we will omit the superscript  $H$  for  $B$ ) and  $H_q$  is the Hermite polynomial of degree  $q \geq 1$  given by  $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} (e^{-\frac{x^2}{2}})$ . The sequence  $V_n$  behaves as follows (see e.g. [9], Theorem 1; the result is also recalled in Section 3 of our paper): if  $0 < H < 1 - \frac{1}{2q}$ , a central limit theorem holds for the renormalized sequence  $Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}$  while if  $1 - \frac{1}{2q} < H < 1$ , the sequence  $Z_n^{(2)} = \frac{V_n}{c_{2,q,H}n^{1-q(1-H)}}$  converges in  $L^2(\Omega)$  to a Hermite random variable of order  $q$  (see Section 2 for the definition of the Hermite random variable and Section 3 for a rigorous statement concerning the convergence of  $V_n$ ). Here  $c_{1,q,H}, c_{2,q,H}$  are explicit positive constants depending on  $q$  and  $H$ .

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer's results are strongly related to the independence of the random variables  $X_1, X_2, \dots$ . In our case the variables are correlated. Indeed, for any  $k, l \geq 1$  we have  $\mathbf{E}(H_q(B_{k+1} - B_k)H_q(B_{l+1} - B_l)) = \frac{1}{(q!)^2} \rho_H(k - l)$  where the correlation function is  $\rho_H(k) = \frac{1}{2}((k+1)^{2H} + (k-1)^{2H} - 2k^{2H})$  which is not equal to zero unless  $H = \frac{1}{2}$  (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [10] via Stein's method and Malliavin calculus. Concretely, we study in this paper the behavior as  $\varepsilon \rightarrow 0$  of the quantities

$$\sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right), \quad (4)$$

and

$$\sum_{n \geq 1} P(V_n > \varepsilon n) = \sum_{n \geq 1} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right), \quad (5)$$

if  $0 < H < 1 - \frac{1}{2q}$  and of

$$\sum_{n \geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \quad (6)$$

and

$$\sum_{n \geq 1} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n \geq 1} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \quad (7)$$

if  $1 - \frac{1}{2q} < H < 1$ . The basic idea in our proofs is that, if we replace  $Z_n^{(1)}$  and  $Z_n^{(2)}$  by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as  $\varepsilon \rightarrow 0$  can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of  $Z_n^{(1)}, Z_n^{(2)}$  and the tail probabilities of their limits. To this end, we will use the estimates obtained in [2], [10] via Malliavin calculus and we are able to prove that this difference converges to zero in all cases. We obtain that, as  $\varepsilon \rightarrow 0$ , the quantities (4) and (6) are of order of  $\log \varepsilon$  while the functions (5) and (7) are of order of  $\varepsilon^2$  and  $\varepsilon^{1-q(1-H)}$  respectively.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic analysis on Wiener chaos. In Section 3 we prove the Spitzer's theorem for the variations of the fractional Brownian motion while Section 4 is devoted to the Hsu-Robbins theorem for this sequence.

Throughout the paper we will denote by  $c$  a generic strictly positive constant which may vary from line to line (and even on the same line).

## 2 Preliminaries

Let  $(W_t)_{t \in [0,1]}$  be a classical Wiener process on a standard Wiener space  $(\Omega, \mathcal{F}, \mathbf{P})$ . If  $f \in L^2([0, 1]^n)$  with  $n \geq 1$  integer, we introduce the multiple Wiener-Itô integral of  $f$  with respect to  $W$ . The basic reference is [11].

Let  $f \in \mathcal{S}_m$  be an elementary function with  $m$  variables that can be written as

$$f = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} 1_{A_{i_1} \times \dots \times A_{i_m}}$$

where the coefficients satisfy  $c_{i_1, \dots, i_m} = 0$  if two indices  $i_k$  and  $i_l$  are equal and the sets  $A_i \in \mathcal{B}([0, 1])$  are disjoint. For such a step function  $f$  we define

$$I_m(f) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} W(A_{i_1}) \dots W(A_{i_m})$$

where we put  $W(A) = \int_0^1 1_A(s) dW_s$  if  $A \in \mathcal{B}([0, 1])$ . It can be seen that the mapping  $I_n$  constructed above from  $\mathcal{S}_m$  to  $L^2(\Omega)$  is an isometry on  $\mathcal{S}_m$ , i.e.

$$\mathbf{E}[I_n(f)I_m(g)] = n! \langle f, g \rangle_{L^2([0,1]^n)} \text{ if } m = n \quad (8)$$

and

$$\mathbf{E}[I_n(f)I_m(g)] = 0 \text{ if } m \neq n.$$

Since the set  $\mathcal{S}_n$  is dense in  $L^2([0, 1]^n)$  for every  $n \geq 1$  the mapping  $I_n$  can be extended to an isometry from  $L^2([0, 1]^n)$  to  $L^2(\Omega)$  and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [8], Theorem 4.1)

$$P(|I_n(f)| > u) \leq c \exp \left( \left( \frac{-cu}{\sigma} \right)^{\frac{2}{n}} \right) \quad (9)$$

for all  $u > 0$ ,  $n \geq 1$ , with  $\sigma = \|f\|_{L^2([0,1]^n)}$ .

The Hermite random variable of order  $q \geq 1$  that appears as limit in Theorem 1, point ii. is defined as (see [9])

$$Z = d(q, H) I_q(L) \quad (10)$$

where the kernel  $L \in L^2([0, 1]^q)$  is given by

$$L(y_1, \dots, y_q) = \int_{y_1 \vee \dots \vee y_q}^1 \partial_1 K^H(u, y_1) \dots \partial_1 K^H(u, y_q) du.$$

The constant  $d(q, H)$  is a positive normalizing constant that guarantees that  $\mathbf{E}Z^2 = 1$  and  $K^H$  is the standard kernel of the fractional Brownian motion (see [11], Section 5). We will not need the explicit expression of this kernel. Note that the case  $q = 1$  corresponds to the fractional Brownian motion and the case  $q = 2$  corresponds to the Rosenblatt process.

### 3 Spitzer's theorem

Let us start by recalling the following result on the convergence of the sequence  $V_n$  (3) (see [9], Theorem 1).

**Theorem 1** *Let  $q \geq 2$  an integer and let  $(B_t)_{t \geq 0}$  a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ . Then, with some explicit positive constants  $c_{1,q,H}, c_{2,q,H}$  depending only on  $q$  and  $H$  we have*

i. *If  $0 < H < 1 - \frac{1}{2q}$  then*

$$\frac{V_n}{c_{1,q,H} \sqrt{n}} \xrightarrow[n \rightarrow \infty]{\text{Law}} N(0, 1) \quad (11)$$

ii. If  $1 - \frac{1}{2q} < H < 1$  then

$$\frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \xrightarrow[n \rightarrow \infty]{L^2} Z \quad (12)$$

where  $Z$  is a Hermite random variable given by (10).

In the case  $H = 1 - \frac{1}{2q}$  the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work.

We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H} \sqrt{n}}, \quad Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \quad (13)$$

with the constants  $c_{1,q,H}, c_{2,q,H}$  from Theorem 1.

Let us denote, for every  $\varepsilon > 0$ ,

$$f_1(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n \geq 1} \frac{1}{n} P(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}) \quad (14)$$

and

$$f_2(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n^{2-2q(1-H)}) = \sum_{n \geq 1} \frac{1}{n} P(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}) \quad (15)$$

**Remark 1** It is natural to consider the tail probability of order  $n^{2-2q(1-H)}$  in (15) because the  $L^2$  norm of the sequence  $V_n$  is in this case of order  $n^{1-q(1-H)}$ .

We are interested to study the behavior of  $f_i(\varepsilon)$  ( $i = 1, 2$ ) as  $\varepsilon \rightarrow 0$ . For a given random variable  $X$ , we set  $\Phi_X(z) = 1 - P(X < z) + P(X < -z)$ .

The first lemma gives the asymptotics of the functions  $f_i(\varepsilon)$  as  $\varepsilon \rightarrow 0$  when  $Z_n^{(i)}$  are replaced by their limits.

**Lemma 1** Consider  $c > 0$ .

i. Let  $Z^{(1)}$  be a standard normal random variable. Then as

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon \sqrt{n}) \xrightarrow{\varepsilon \rightarrow 0} 2.$$

ii. Let  $Z^{(2)}$  be a Hermite random variable of order  $q$  given by (10). Then, for any integer  $q \geq 1$

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{1-q(1-H)}.$$

**Proof:** The case when  $Z^{(1)}$  follows the standard normal law is hidden in [12]. We will give the ideas of the proof. We can write (see [12])

$$\sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx - \frac{1}{2} \Phi_{Z^{(1)}}(c\varepsilon) - \int_1^\infty P_1(x) d \left[ \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) \right].$$

with  $P_1(x) = [x] - x + \frac{1}{2}$ . Clearly as  $\varepsilon \rightarrow 0$ ,  $\frac{1}{\log \varepsilon} \Phi_{Z^{(1)}}(c\varepsilon) \rightarrow 0$  because  $\Phi_{Z^{(1)}}$  is a bounded function and concerning the last term it is also trivial to observe that

$$\begin{aligned} & \frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d \left[ \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) \right] \\ &= \frac{1}{-\log c\varepsilon} \left( - \int_1^\infty P_1(x) \left( \frac{1}{x^2} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx + c\varepsilon \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{x} \Phi'_{Z^{(1)}}(\varepsilon\sqrt{x}) \right) dx \right) \rightarrow_{\varepsilon \rightarrow 0} 0 \end{aligned}$$

since  $\Phi_{Z^{(1)}}$  and  $\Phi'_{Z^{(1)}}$  are bounded. Therefore the asymptotics of the function  $f_1(\varepsilon)$  as  $\varepsilon \rightarrow 0$  will be given by  $\int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx$ . By making the change of variables  $c\varepsilon\sqrt{x} = y$ , we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} 2 \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(1)}}(y) dy = \lim_{\varepsilon \rightarrow 0} 2 \Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \left( \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx - \int_1^\infty P_1(x) d \left[ \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) \right] \right) \end{aligned}$$

By making the change of variables  $c\varepsilon x^{1-q(1-H)} = y$  we will obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(2)}}(y) dy = \lim_{\varepsilon \rightarrow 0} \frac{1}{1-q(1-H)} \Phi_{Z^{(2)}}(c\varepsilon) = \frac{1}{1-q(1-H)} \end{aligned}$$

where we used the fact that  $\Phi_{Z^{(2)}}(y) \leq y^{-2} \mathbf{E}|Z^{(2)}|^2$  and so  $\lim_{y \rightarrow \infty} \log y \Phi_{Z^{(2)}}(y) = 0$ .

It remains to show that  $\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d \left[ \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) \right]$  converges to zero as  $\varepsilon$  tends to 0 (note that actually it follows from a result by [1] that a Hermite random variable has a density, but we don't need it explicitly, we only use the fact that  $\Phi_{Z^{(2)}}$  is differentiable almost everywhere). This is equal to

$$\begin{aligned} & \lim_{\varepsilon} \frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) c\varepsilon (1-q(1-H)) x^{-q(1-H)-1} \Phi'_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\ &= c \frac{\varepsilon}{-\log \varepsilon} (c\varepsilon)^{\frac{q(1-H)}{1-q(1-H)}} \int_{c\varepsilon}^\infty P_1 \left( \left( \frac{y}{c\varepsilon} \right)^{\frac{1}{1-q(1-H)}} \right) \Phi'_{Z^{(2)}}(y) y^{-\frac{1}{1-q(1-H)}} dy \\ &\leq c \frac{1}{-\log \varepsilon} \int_{c\varepsilon}^\infty P_1 \left( \left( \frac{1}{c\varepsilon} \right)^{\frac{1}{1-q(1-H)}} \right) \Phi'_{Z^{(2)}}(y) dy \end{aligned}$$

which clearly goes to zero since  $P_1$  is bounded and  $\int_0^\infty \Phi'_{Z^{(2)}}(y)dy = 1$ . ■

The next result estimates the limit of the difference between the functions  $f_i(\varepsilon)$  given by (14), (15) and the sequence in Lemma 1.

**Proposition 1** *Let  $q \geq 2$  and  $c > 0$ .*

- i. *If  $H < 1 - \frac{1}{2q}$ , let  $Z_n^{(1)}$  be given by (13) and let  $Z^{(1)}$  be standard normal random variable. Then it holds*

$$\frac{1}{-\log c\varepsilon} \left[ \sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(1)}| > c\varepsilon\sqrt{n}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(1)}| > c\varepsilon\sqrt{n}) \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

- ii. *Let  $Z^{(2)}$  be a Hermite random variable of order  $q \geq 2$  and  $H > 1 - \frac{1}{2q}$ . Then*

$$\frac{1}{-\log c\varepsilon} \left[ \sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}) \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

**Proof:** Let us start with the point i. Assume  $H < 1 - \frac{1}{2q}$ . We can write

$$\begin{aligned} & \sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(1)}| > c\varepsilon\sqrt{n}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(1)}| > c\varepsilon\sqrt{n}) \\ &= \sum_{n \geq 1} \frac{1}{n} \left[ P(Z_n^{(1)} > c\varepsilon\sqrt{n}) - P(Z^{(1)} > c\varepsilon\sqrt{n}) \right] + \sum_{n \geq 1} \left[ \frac{1}{n} P(Z_n^{(1)} < -c\varepsilon\sqrt{n}) - P(Z^{(1)} < -c\varepsilon\sqrt{n}) \right] \\ &\leq 2 \sum_{n \geq 1} \frac{1}{n} \sup_{x \in \mathbb{R}} \left| P(Z_n^{(1)} > x) - P(Z^{(1)} > x) \right|. \end{aligned}$$

It follows from [10], Theorem 4.1 that

$$\sup_{x \in \mathbb{R}} \left| P(Z_n^{(1)} > x) - P(Z^{(1)} > x) \right| \leq c \begin{cases} \frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ n^{H-1}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ n^{qH-q+\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}). \end{cases} \quad (16)$$

and this implies that

$$\sum_{n \geq 1} \frac{1}{n} \sup_{x \in \mathbb{R}} \left| P(Z_n^{(i)} > x) - P(Z^{(i)} > x) \right| \leq c \begin{cases} \sum_{n \geq 1} \frac{1}{n\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ \sum_{n \geq 1} n^{H-2}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ \sum_{n \geq 1} n^{qH-q-\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}). \end{cases} \quad (17)$$

and the last sums are finite (for the last one we use  $H < 1 - \frac{1}{2q}$ ). The conclusion follows.

Concerning the point ii. (the case  $H > 1 - \frac{1}{2q}$ ), by using a result in Proposition 3.1 of [2] we have

$$\sup_{x \in \mathbb{R}} \left| P\left(Z_n^{(i)} > x\right) - P\left(Z^{(i)} > x\right) \right| \leq c \left( \mathbf{E} \left| Z_n^{(2)} - Z^{(2)} \right|^2 \right)^{\frac{1}{2q}} \leq c n^{1 - \frac{1}{2q} - H} \quad (18)$$

and as a consequence

$$\sum_{n \geq 1} \frac{1}{n} P\left(|Z_n^{(2)}| > c \varepsilon n^{1-q(1-H)}\right) - \sum_{n \geq 1} \frac{1}{n} P\left(|Z^{(2)}| > c \varepsilon n^{1-q(1-H)}\right) \leq c \sum_{n \geq 1} n^{-\frac{1}{2q}-H}$$

and the above series is convergent because  $H > 1 - \frac{1}{2q}$ . ■

We state now the Spitzer's theorem for the variations of the fractional Brownian motion.

**Theorem 2** *Let  $f_1, f_2$  be given by (14), (15) and the constants  $c_{1,q,H}, c_{2,q,H}$  be those from Theorem 1.*

i. *If  $0 < H < 1 - \frac{1}{2q}$  then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\log(c_{1,H,q}^{-1} \varepsilon)} f_1(\varepsilon) = 2.$$

ii. *If  $1 > H > 1 - \frac{1}{2q}$  then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\log(c_{2,H,q}^{-1} \varepsilon)} f_2(\varepsilon) = \frac{1}{1 - q(1 - H)}.$$

**Proof:** It is a consequence of Lemma 1 and Proposition 1. ■

**Remark 2** *Concerning the case  $H = 1 - \frac{1}{2q}$ , note that the correct normalization of  $V_n$  (3) is  $\frac{1}{(\log n) \sqrt{n}}$ . Because of the appearance of the term  $\log n$  our approach is not directly applicable to this case.*

## 4 Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every  $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n \geq 1} P(|V_n| > \varepsilon n) \quad (19)$$



if  $H < 1 - \frac{1}{2q}$  and by

$$g_2(\varepsilon) = \sum_{n \geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \quad (20)$$

if  $H > 1 - \frac{1}{2q}$ . and we estimate the behavior of the functions  $g_i(\varepsilon)$  as  $\varepsilon \rightarrow 0$ . Note that we can write

$$g_1(\varepsilon) = \sum_{n \geq 1} P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right), \quad g_2(\varepsilon) = \sum_{n \geq 1} P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right)$$

with  $Z_n^{(1)}, Z_n^{(2)}$  given by (13).

We decompose it as: for  $H < 1 - \frac{1}{2q}$

$$\begin{aligned} g_1(\varepsilon) &= \sum_{n \geq 1} P\left(|Z^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) \\ &+ \sum_{n \geq 1} \left[ P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) - P\left(|Z^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) \right]. \end{aligned}$$

and for  $H > 1 - \frac{1}{2q}$

$$\begin{aligned} g_2(\varepsilon) &= \sum_{n \geq 1} P\left(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\right) \\ &+ \sum_{n \geq 1} \left[ P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \right]. \end{aligned}$$

We start again by consider the situation when  $Z_n^{(i)}$  are replaced by their limits.

**Lemma 2** *i. Let  $Z^{(1)}$  be a standard normal random variable. Then*

$$\lim_{\varepsilon \rightarrow 0} (c\varepsilon)^2 \sum_{n \geq 1} P\left(|Z^{(1)}| > c\varepsilon \sqrt{n}\right) = 1.$$

*ii. Let  $Z^{(2)}$  be a Hermite random variable with  $H > 1 - \frac{1}{2q}$ . Then*

$$\lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) = \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

**Proof:** The part i. is a consequence of the result of Heyde [5]. Indeed take  $X_i \sim N(0, 1)$  in (1). Concerning part ii. we can write

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ &= \lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \left[ \int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx - \int_1^\infty P_1(x) d\left[\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right] \right] \\ &:= \lim_{\varepsilon \rightarrow 0} (A(\varepsilon) + B(\varepsilon)) \end{aligned}$$

with  $P_1(x) = [x] - x + \frac{1}{2}$ . Moreover

$$\begin{aligned} A(\varepsilon) &= (c\varepsilon)^{\frac{1}{1-q(1-H)}} \int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\ &= \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}-1} dy. \end{aligned}$$

Since  $\Phi_{Z^{(2)}}(y) \leq y^{-2}$  we have  $\Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} \rightarrow_{y \rightarrow \infty} 0$  and therefore

$$A(\varepsilon) = -\Phi_{Z^{(2)}}(c\varepsilon) (c\varepsilon)^{\frac{1}{1-q(1-H)}} - \int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} dy$$

where the first terms goes to zero and the second to  $\mathbf{E} |Z^{(2)}|^{\frac{1}{1-q(1-H)}}$ . The proof that the term  $B(\varepsilon)$  converges to zero is similar to the proof of Lemma 2, point ii.  $\blacksquare$

**Remark 3** *The Hermite random variable has moments of all orders (in particular the moment of order  $\frac{1}{1-q(1-H)}$  exists) since it is the value at time 1 of a selfsimilar process with stationary increments.*

**Proposition 2** *i. Let  $H < 1 - \frac{1}{2q}$  and let  $Z_n^{(1)}$  be given by (13). Let also  $Z^{(1)}$  be a standard normal random variable. Then*

$$(c\varepsilon)^2 \sum_{n \geq 1} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \right] \rightarrow_{\varepsilon \rightarrow 0} 0$$

*ii. Let  $H > 1 - \frac{1}{2q}$  and let  $Z_n^{(2)}$  be given by (13). Let  $Z^{(2)}$  be a Hermite random variable. Then*

$$(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \left[ P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) \right] \rightarrow_{\varepsilon \rightarrow 0} 0.$$

**Remark 4** *Note that the bounds (16), (18) does not help here because the series that appear after their use are not convergent.*

**Proof of Proposition 2:** *Case  $H < 1 - \frac{1}{2q}$ . We have, for some  $\beta > 0$  to be chosen later,*

$$\begin{aligned} & \varepsilon^2 \sum_{n \geq 1} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \right] \\ &= \varepsilon^2 \sum_{n=1}^{\lfloor \varepsilon^{-\beta} \rfloor} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \right] \\ & \quad + \varepsilon^2 \sum_{n > \lfloor \varepsilon^{-\beta} \rfloor} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \right] \\ &:= I_1(\varepsilon) + J_1(\varepsilon). \end{aligned}$$

Consider first the situation when  $H \in (0, \frac{1}{2}]$ . Let us choose a real number  $\beta$  such that  $2 < \beta < 4$ . By using (16),

$$I_1(\varepsilon) \leq c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \leq c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \rightarrow_{\varepsilon \rightarrow 0} 0$$

since  $\beta < 4$ . Next, by using the bound for the tail probabilities of multiple integrals and since  $\mathbf{E} \left| Z_n^{(1)} \right|^2$  converges to 1 as  $n \rightarrow \infty$

$$\begin{aligned} J_1(\varepsilon) &= \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P \left( Z_n^{(1)} > c\varepsilon \sqrt{n} \right) \leq c\varepsilon^{-2} \sum_{n > [\varepsilon^{-\beta}]} \exp \left( \frac{-c\varepsilon \sqrt{n}}{\left( \mathbf{E} \left| Z_n^{(1)} \right|^2 \right)^{\frac{1}{2}}} \right)^{\frac{2}{q}} \\ &\leq \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} \exp \left( \left( -cn^{-\frac{1}{\beta}} \sqrt{n} \right)^{\frac{2}{q}} \right) \end{aligned}$$

and since converges to zero for  $\beta > 2$ . The same argument shows that  $\varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P \left( Z^{(1)} > c\varepsilon \sqrt{n} \right)$  converges to zero.

The case when  $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$  can be obtained by taking  $2 < \beta < \frac{2}{H}$  (it is possible since  $H < 1$ ) while in the case  $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$  we have to choose  $2 < \beta < \frac{2}{qH - q + \frac{3}{2}}$  (which is possible because  $H < 1 - \frac{1}{2q}$ !).

*Case  $H > 1 - \frac{1}{2q}$ .* We have, with some suitable  $\beta > 0$

$$\begin{aligned} &\varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \left[ P \left( |Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P \left( |Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right] \\ &= \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[ P \left( |Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P \left( |Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right] \\ &\quad + \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \geq [\varepsilon^{-\beta}]} \left[ P \left( |Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P \left( |Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right] \\ &:= I_2(\varepsilon) + J_2(\varepsilon). \end{aligned}$$

Choose  $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$  (again, this is always possible when  $H > 1 - \frac{1}{2q}$ !). Then

$$I_2(\varepsilon) \leq c\varepsilon^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \rightarrow_{\varepsilon \rightarrow 0} 0$$

and by (9)

$$J_2(\varepsilon) \leq c \sum_{n > [\varepsilon^{-\beta}]} \exp \left( \left( \frac{-c\varepsilon n^{1-q(1-H)}}{\left( \mathbf{E} |Z_n^{(2)}|^2 \right)^{\frac{1}{2}}} \right)^{\frac{2}{q}} \right) \leq c \sum_{n > [\varepsilon^{-\beta}]} \exp \left( cn^{-\frac{1}{\beta}} n^{1-q(1-H)} \right)^{\frac{2}{q}} \xrightarrow{\varepsilon \rightarrow 0} 0$$

■

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

**Theorem 3** *Let  $q \geq 2$  and let  $c_{1,q,H}, c_{2,q,H}$  be the constants from Theorem 1. Let  $Z^{(1)}$  be a standard normal random variable,  $Z^{(2)}$  a Hermite random variable of order  $q \geq 2$  and let  $g_1, g_2$  be given by (19) and (20). Then*

- i. *If  $0 < H < 1 - \frac{1}{2q}$ , we have  $(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 1 = \mathbf{E} Z^{(1)}$ .*
- ii. *If  $1 - \frac{1}{2q} < H < 1$  we have  $(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} \mathbf{E} |Z^{(2)}|^{\frac{1}{1-q(1-H)}}$ .*

**Remark 5** *In the case  $H = \frac{1}{2}$  we retrieve the result (1) of [5]. The case  $q = 1$  is trivial, because in this case, since  $V_n = B_n$  and  $\mathbf{E} V_n^2 = n^{2H}$ , we obtain the following (by applying Lemma 1 and 2 with  $q = 1$ )*

$$\frac{1}{\log \varepsilon} \sum_{n \geq 1} \frac{1}{n} P(|V_n| > \varepsilon n^{2H}) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{H}$$

and

$$\varepsilon^2 \sum_{n \geq 1} P(|V_n| > \varepsilon n^{2H}) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} |Z^{(1)}|^{\frac{1}{H}}.$$

**Remark 6** *Let  $(\varepsilon_i)_{i \in \mathbb{Z}}$  be a sequence of i.i.d. centered random variable with finite variance and let  $(a_i)_{i \geq 1}$  a square summable real sequence. Define  $X_n = \sum_{i \geq 1} a_i \varepsilon_{n-i}$ . Then the sequence  $S_N = \sum_{n=1}^N [K(X_n) - \mathbf{E} K(X_n)]$  satisfies a central limit theorem or a non-central limit theorem according to the properties of the measurable function  $K$  (see [6] or [14]). We think that our tools can be applied to investigate the tail probabilities of the sequence  $S_N$  in the spirit of [5] or [12] at least the in particular cases (for example, when  $\varepsilon_i$  represents the increment  $W_{i+1} - W_i$  of a Wiener process because in this case  $\varepsilon_i$  can be written as a multiple integral of order one and  $X_n$  can be decomposed into a sum of multiple integrals. We thank the referee for mentioning the references [6] and [14].*

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